

8.1 – General Linear Transformations

Definition: (analogous to Theorem 1.8.2)

If $T : V \rightarrow W$ is a mapping from a vector space V to a vector space W , then T is called a **linear transformation** from V to W if the following two properties hold for all vectors \mathbf{u} and \mathbf{v} in V and for all scalars k :

- a) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)
- b) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V .

Theorem 8.1.1 (analogous to Theorem 1.8.1)

If $T : V \rightarrow W$ is a linear transformation, then:

- a) $T(\mathbf{0}) = \mathbf{0}$.
 - b) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all \mathbf{u} and \mathbf{v} in V .
 - c) $T(-\mathbf{v}) = -T(\mathbf{v})$ for all \mathbf{v} in V .
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Examples of Linear transformations

- $T : V \rightarrow W$ defined by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (the zero transformation)
- $I : V \rightarrow V$ defined by $I(\mathbf{v}) = \mathbf{v}$ (the identity operator)
- $T : V \rightarrow V$ defined by $T(\mathbf{x}) = c\mathbf{x}$ (contraction if $0 < c < 1$ and dilation if $c > 1$)
- $T : V \rightarrow R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ (inner product of \mathbf{x} with \mathbf{x}_0) [We'll see this in Ch. 6]
- $T : M_{nn} \rightarrow M_{nn}$ defined by $T(A) = A^T$

- $T : V \rightarrow R^n$ defined by $T(f) = (f(x_1), f(x_2), \dots, f(x_n))$, where V is a subspace of $F(-\infty, \infty)$, and x_1, x_2, \dots, x_n is a sequence of real numbers (evaluation transformation)
- $D : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$ defined by $D(f) = f'(x)$ (differentiation)
- $J : C(-\infty, \infty) \rightarrow C^1(-\infty, \infty)$ defined by $J(f) = \int_0^x f(t) dt$ (integration)

Theorem 8.1.2 Let $T : V \rightarrow W$ be a linear transformation, for which the vector space V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , then the image of any vector \mathbf{v} in V can be expressed as

$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \dots + c_n T(\mathbf{v}_n)$ where c_1, c_2, \dots, c_n are the coefficients required to express \mathbf{v} as a linear combination of the vectors in the basis S .

#20 Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2 , where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T : R^2 \rightarrow R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1, 2, 0)$ and $T(\mathbf{v}_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2, -3)$.



Definition: (analogous to definitions seen in Section 4.2 (**kernel**) and Section 1.8 (**range**); also related to Definition 2 of Section 4.8 (**null space**) and, by Theorem 4.8.1, column space)

If $T : V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

#7 Determine whether the mapping T is a linear transformation, and if so, find its kernel.

$T : P_2 \rightarrow P_2$, where

a. $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x + 1) + a_2(x + 1)^2$

b. $T(a_0 + a_1x + a_2x^2) = (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$

#10 Let $T : P_2 \rightarrow P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(T)$?

- a. x^2
 - b. 0
 - c. $1 + x$
 - d. $-x$
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#11 Let $T : P_2 \rightarrow P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

- a. $x + x^2$
 - b. $1 + x$
 - c. $3 - x^2$
 - d. $-x$
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